One double inequality for a right triangle.

https://www.linkedin.com/feed/update/urn:li:activity:6635435271313858560

Let a, b, c be side lengths of a right triangle with hypotenuse c. Show that

$$3 < \frac{c^3 - a^3 - b^3}{c(c-a)(c-b)} \le \sqrt{2} + 2.$$

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Let $t := ab$. Then assuming $c = 1$ (due to homogeneity of the expression) we obtain taking in account that $a^2 + b^2 = 1$ that

$$\frac{c^3 - a^3 - b^3}{c(c-a)(c-b)} = \frac{1 - (a+b)(a^2 + b^2 - ab)}{(1-a)(1-b)} = \frac{1 - (a+b)(1-ab)}{1 - (a+b) + ab} = \frac{1 - (1-t)\sqrt{1+2t}}{1 - \sqrt{1+2t} + t} =$$

$$\frac{1 - \sqrt{1+2t} + t\sqrt{1+2t}}{1 - \sqrt{1+2t} + t} = 1 + \frac{t\left(\sqrt{1+2t} - 1\right)}{1 + t - \sqrt{1+2t}} = 1 + \frac{t(1+2t-1)\left(1 + t + \sqrt{1+2t}\right)}{(1+t)^2 - (1+2t)\sqrt{1+2t} + 1} =$$

$$1 + \frac{2t^2\left(1 + t + \sqrt{1+2t}\right)}{t^2\sqrt{1+2t} + 1} = 1 + \frac{2\left(1 + t + \sqrt{1+2t}\right)}{\sqrt{1+2t} + 1} = 3 + 2 \bullet \frac{t}{\sqrt{1+2t} + 1}.$$

Hence, $3 < \dfrac{c^3 - a^3 - b^3}{c(c-a)(c-b)}$ and since $t = ab \le \dfrac{a^2 + b^2}{2} = \dfrac{1}{2}$ we obtain*

$$\frac{c^3 - a^3 - b^3}{c(c-a)(c-b)} = 3 + \frac{2t}{\sqrt{1+2t} + 1} \le 3 + \frac{2 \bullet 1/2}{\sqrt{1 + 2 \bullet 1/2} + 1} = 3 + \left(\sqrt{2} - 1\right) = \sqrt{2} + 2.$$

* $\dfrac{t}{\sqrt{1+2t} + 1} = \dfrac{1}{\sqrt{\dfrac{1}{t^2} + \dfrac{2}{t}} + \dfrac{1}{t}}$ increase by $t \in (0, \infty)$ (because $\sqrt{\dfrac{1}{t^2} + \dfrac{2}{t}} + \dfrac{1}{t}$ decrease by $t > 0$).